The Electronic Journal of Mathematics and Technology, Volume 15, Number 1, ISSN 1933-2823

3D Locus Problems Of Lines Passing Through A Fixed Point

Wei-Chi Yang
wyang@radford.edu
Department of Mathematics and Statistics
Radford University
VA 24142
USA

Antonio Morante
amorante@fciencias.uaslp.mx
Facultad de Ciencias
UASLP
San Luis Potosí (SLP) CP 78290
México

Abstract

We apply the use of Vieta's formula and the results of locus curves from 2D in [9] to find the locus surfaces on some quadratic surfaces. We explore the locus surfaces E, which can be obtained by rotating a proper 2D locus curve around a rotating axis. Moreover, we discuss how the Vieta's formulas can be applied in special surfaces when solving higher degree polynomial equations.

1 Introduction

The problems discussed in this paper are extensions from those 2D problems discussed in [9] to corresponding 3D problems. In this article we explore the following

Main problem: *We are given a fixed point $A \in \mathbb{R}^3$ and a point C on a surface Σ. We let the line l pass through A and C and intersect a well-defined D on Σ, we want to determine the locus surface generated by the point E, lying on CD, which satisfies $\overrightarrow{ED} = s\overrightarrow{CD}$, where s is a real number parameter.*

Activities explored in this paper can be beneficial to readers who have knowledge in multi-variable calculus. In Section 2, we investigate how the Vieta's formulas, we adopted in finding the locus curves in 2D (see [9]), can be used in selected quadric surfaces. In Section 3, we start with a locus curve $d(t)$ of a given curve $c(t)$, where $t \in [a, b]$ and we shall see how we can find the corresponding locus surface. We first rotate the curve $c(t)$ around a proper axis to get the corresponding surface, then we shall see that the corresponding locus surface can be obtained by rotating $d(t)$ around a proper axis too. In Section 4, we explore surfaces that are central symmetric, surfaces that are symmetric to the origin, and their respective locus surfaces can be found easily. Consequently, it leads us to investigate some special higher degree surfaces where either Vieta's formulas can be applied or not needed when finding the locus surfaces.

2 Quadric Surfaces: The fixed point A at arbitrary location

We recall that to the main thrust of finding the locus curve for a given 2D curve, in [9], is to apply the Vieta's formulas, on both the implicit and parametric equations for that given curve. In this section, we follow these ideas in order to find the locus surfaces when the given surfaces are quadrics. We also recall that for a real polynomial of degree n, say $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, having roots $r_1, r_2, \ldots, r_n$, then Vieta's formulas are:

$$\begin{aligned} r_1 + r_2 + \cdots + + r_{n-1} + r_n &= -\frac{a_{n-1}}{a_n}, \\ (r_1r_2 + \cdots + r_1r_n) + (r_2r_3 + \cdots + r_2r_n) + (r_{n-1}r_n) &= \frac{a_{n-2}}{a_n}, \\ &\vdots \\ r_1r_2\cdots r_{n-1}r_n &= (-1)^n \frac{a_0}{a_n}. \end{aligned}$$

2.1 The ellipsoid

Let $A = (x_0, y_0, z_0)$. Consider the ellipsoid

$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\} \tag{1}$$

Using the standard parametrization for Σ, we can represent a generic point C on Σ as

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} a\cos(u)\sin(v) \\ b\sin(u)\sin(v) \\ c\cos(v) \end{bmatrix} \tag{2}$$

where $u \in [0, 2\pi]$ is "longitude" and $v \in [0, \pi]$ is "colatitude". In order to calculate the coordinates of point $D = (x, y, z)$ (which is different from C), as the intersection between the quadric Σ and the line l passing through A and C, we make use of the parametric equation of line l as follows:

$$\begin{aligned} x - x_0 &= \lambda(\hat{x} - x_0), \\ y - y_0 &= \lambda(\hat{y} - y_0), \\ z - z_0 &= \lambda(\hat{z} - z_0). \end{aligned}$$

Hence, we obtain

$$\frac{y - y_0}{x - x_0} = \frac{\hat{y} - y_0}{\hat{x} - x_0}, \tag{3}$$

$$\frac{z - z_0}{x - x_0} = \frac{\hat{z} - z_0}{\hat{x} - x_0}. \tag{4}$$

By substituting (2) into equations (3) and (4), we get some expressions for the left hand side in (3) and (4), allowing us to define two auxiliary functions, namely

$$k(u,v) = \frac{b\sin(u)\sin(v) - y_0}{a\cos(u)\sin(v) - x_0}, \tag{5}$$

$$m(u,v) = \frac{c\cos(v) - z_0}{a\cos(u)\sin(v) - x_0}. \tag{6}$$

Since both intersection points, C and D, satisfy the implicit equation of Σ, we can use (5) and (6) to get the x–coordinate of D, say x_1, by calculating the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0,$$

where

$$a_2 = \frac{a^2b^2m(u,v)^2 + a^2c^2k(u,v)^2 + b^2c^2}{a^2b^2c^2} \tag{7}$$

$$a_1 = \frac{2\big(z_0b^2m(u,v) + y_0c^2k(u,v) - x_0(b^2m(u,v)^2 + c^2k(u,v)^2)\big)}{b^2c^2} \tag{8}$$

$$a_0 = \frac{x_0^2\big(b^2m(u,v)^2 + c^2k(u,v)^2\big) - 2x_0\big(z_0b^2m(u,v) + y_0c^2k(u,v)\big) + y_0^2c^2 + z_0^2b^2 - b^2c^2}{b^2c^2}. \tag{9}$$

It follows from $p(\hat{x}) = 0$ and the Vieta's formulas that

$$x_1 = -\frac{a_1}{a_2} - \hat{x}.$$

It follows from (3) and (4) that

$$y_1 = y_0 + k(x_1 - x_0) \quad \text{and} \quad z_1 = z_0 + m(x_1 - x_0).$$

For a given s, the locus surface generated by point $E = sC + (1-s)D$ is defined as

$$\Delta(u,v) = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1-s)x_1 \\ s\hat{y} + (1-s)y_1 \\ s\hat{z} + (1-s)z_1 \end{bmatrix}.$$

The explicit form of the locus surface Δ can be found in Exploration [S1]. We use the following Example to demonstrate how we find the locus for a particular ellipsoid.

Example 1 *Consider the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

when $a = 5, b = 4,$ and $c = 3$. We let the fixed point $A = (2, -3, 4)$. A few locus traces for the point $E = sC + (1-s)D$, when $s = 2, u \in [0, 2\pi]$, $v = \pi/8, 3\pi/8, 5\pi/8$ and $7\pi/8$ (top to bottom) *are shown in Figure 1(a). The corresponding locus surface generated by point* $E = sC + (1-s)D$ *when* $s = 2$ *is shown in Figure 1(b).*

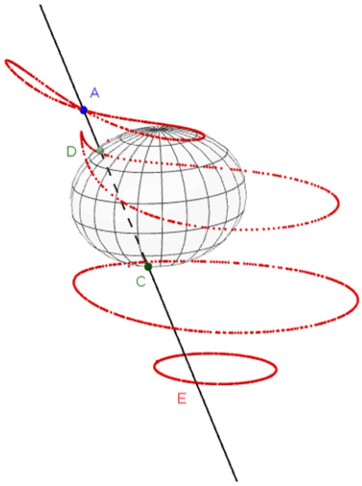

Figure 1(a). Locus traces

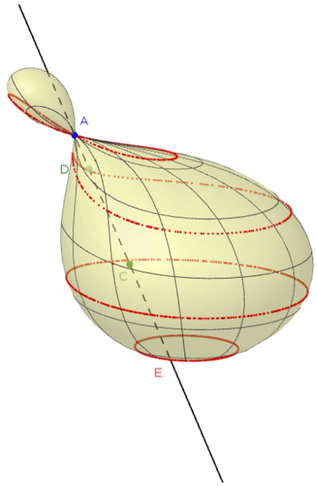

Figure 1(b). Locus surface

We can also explore the locus surface for the ellipsoid for the following scenarios using Netpad (see [5]). We plot the surface when $a = 5, b = 4, c = 3$, $s = 1.7$, *and the fixed point* $A = (2, -3, 4)$ *together with the locus trace when* v *is fixed at* 0.81 *and* u *varies between* 0 *and* 2π *in Figure 2 as follows.*

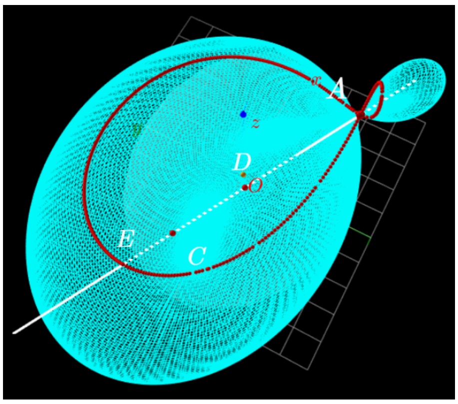

Figure 2. Locus when $a = 5, b = 4, c = 3, s = 1.7$ and $v = 0.81$

Discussions: Let the given surface Σ be compact and convex, and the fixed point A is outside the surface Σ.

1. If $s > 1$, we conjecture that the corresponding locus surface will be containing and tangent to Σ.
2. If $s < 1$, we conjecture that Σ will be inside and tangent to its locus surface.

2.2 Special case: the sphere

When $a = b = c = r$, the ellipsoid (1) becomes the sphere

$$\Sigma_r = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\right\}$$

and the generic point C has the form

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} r\cos(u)\sin(v) \\ r\sin(u)\sin(v) \\ r\cos(v) \end{bmatrix}$$

Now, to find locus surface generated by point D, we use the simplified auxiliary functions

$$k(u, v) = \frac{r\sin(u)\sin(v) - y_0}{r\cos(u)\sin(v) - x_0}$$
$$m(u, v) = \frac{r\cos(v) - z_0}{r\cos(u)\sin(v) - x_0}$$

where

$$a_2 = m(u, v)^2 + k(u, v)^2 + 1 \tag{10}$$
$$a_1 = 2\big(z_0 m(u, v) + y_0 k(u, v) - x_0(m(u, v)^2 + k(u, v)^2)\big) \tag{11}$$
$$a_0 = x_0^2(m(u, v)^2 + k(u, v)^2) - 2x_0\big(z_0 m(u, v) + y_0 k(u, v)\big) + y_0^2 + z_0^2 - r^2. \tag{12}$$

The explicit form of the corresponding locus surface Δ can be found in Exploration [S2].

Example 2 *Consider the sphere*

$$x^2 + y^2 + z^2 = 9.$$

We let the fixed point $A = (2, -3, 4)$, A few locus traces for the point $E = sC + (1 - s)D$, when $s = 1.5, u \in [0, 2\pi], v = \pi/8, 3\pi/8, 5\pi/8$ and $7\pi/8$ (top to bottom) are shown in Figure 3(a). The corresponding locus surface generated by point $E = sC + (1 - s)D$, when $s = 1.5$, is shown *in Figure 3(b).*

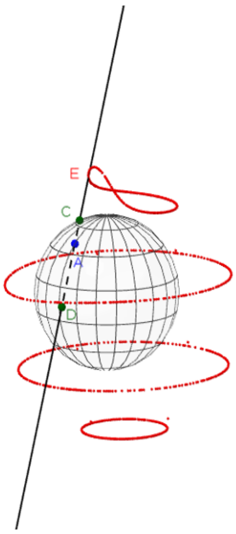

Figure 3(a). Some traces for the locus surface

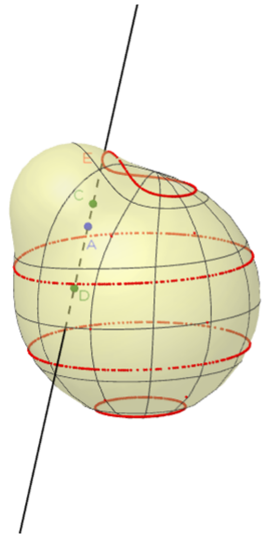

Figure 3(b). Locus surface

Using a simple geometric argument (see Figure 4(a)), it is possible to extend the proof in [9] to show that for A inside Σ_r and $s = 0.5$, the locus surface is the sphere with center at $(\frac{x_0}{2}, \frac{y_0}{2}, \frac{z_0}{2})$ and radius $\frac{1}{2}\sqrt{x_0^2 + y_0^2 + z_0^2}$.

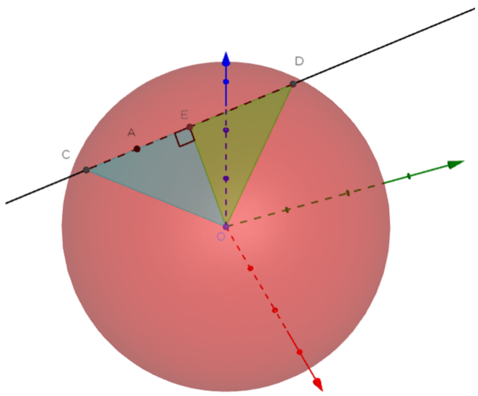

Figure 4(a). Perpendicular bisector of a chord of the sphere

We demonstrate this effect by considering the following

Example 3 *We are given the sphere*

$$x^2 + y^2 + z^2 = 25,$$

and the fixed point $A = (1, -1, 2)$*. As we expected, the locus surface for* $s = 0.5$ *is a sphere with center at* $0.5 \cdot (1, -1, 2)$ *and radius* $0.5\sqrt{1^2 + (-1)^2 + 2^2}$*. We show a few locus traces for the point* $E = sC + (1 - s)D$*, when* $s = 0.5, u \in [0, 2\pi]$*,* $v = \pi/8, 3\pi/8, 5\pi/8$ *and* $7\pi/8$ *(top to bottom) in Figure 4(b). The locus surface generated by point* $E = sC + (1 - s)D$*, when parameter* $s = 0.5$*, is shown in Figure 4(c).*

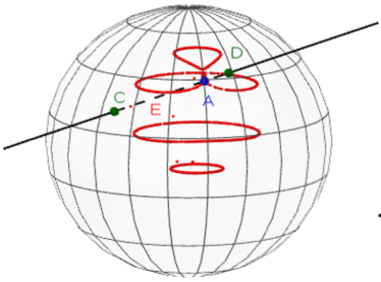

Figure 4(b). Some traces of the locus surface

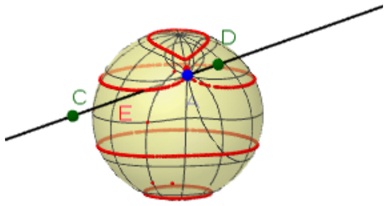

Figure 4(c). Local surface

To explore the locus surface and reproduce animations of the locus traces shown in this section, see Exploration [S3].

3 Rotation Surfaces: The fixed point A is on the rotation axis

In this section, we shall explore the following scenario: Suppose, as was done in [9], that we have found the locus curve $d(t)$ of a given curve $c(t)$, where $t \in [a, b]$. If we rotate the curve $c(t)$ around a proper axis to get the corresponding surface of revolution, then the corresponding locus surface can also be obtained by rotating $d(t)$ around a proper axis. We describe this in details here. Let c be a closed curve with parametric equation

$$c(t) = (f(t), g(t)), \quad t \in [0, 2\pi],$$

and suppose point $A' = (x_0, 0)$ is fixed. Let s be a given parameter and suppose that the lines passing through point A' intersect the curve c at points C' and D', respectively. The parametric equation of the locus curve d, generated by the point E' on line $C'D'$, satisfying $\overrightarrow{E'D'} = s\overrightarrow{C'D'}$, is written as follows:

$$d(t) = (f_{s,A'}(t), g_{s,A'}(t)), \quad t \in [0, 2\pi].$$

Now, let Σ be the surface of revolution, obtained by rotating curve c around the x–axis.

Then we have

$$\Sigma(u,v)=\begin{bmatrix} f(u)\\ g(u)\cos v\\ g(u)\sin v\end{bmatrix},$$

where $u\in[0,2\pi],\ v\in[0,\pi]$ and $A=(x_0,0,0)$ is the fixed point. For lines passing through point A, intersecting the surface Σ at points C and D, respectively, we face the problem of calculating the parametric equation of the locus surface Δ, that is generated by the point E lying on the line CD and satisfying $\overrightarrow{ED}=s\overrightarrow{CD}$. We consider the following observation.

Proposition 4 *Δ is the surface of revolution obtained by rotating "curve d" around the x–axis, that is,*

$$\Delta(u,v)=\begin{bmatrix} f_{s,A'}(u)\\ g_{s,A'}(u)\cos(v)\\ g_{s,A'}(u)\sin(v)\end{bmatrix};\quad u\in[0,2\pi],\ v\in[0,\pi].$$

Proof

Let E be a point on the surface Δ. By the way of our construction for Δ , there exist points C and D on surface Σ such that $E=sC+(1-s)D$. Since Σ is the surface of revolution obtained by rotating curve c around the x–axis, there exist angle $\phi\in[0,\pi]$ and points $C'=[c'_x,c'_y,0]$ and $D'=[d'_x,d'_y,0]$ on the curve

$$\Sigma'(t)=\begin{bmatrix} f(t)\\ g(t)\\ 0\end{bmatrix};\quad t\in[0,2\pi]$$

such that C and D are, respectively, the rotations of C' and D' around the x–axis by ϕ. Because rotations in $\mathbb{R}^3$ are isometries, point $E'=sC'+(1-s)D'$ is on locus curve

$$\Delta'(t)=\begin{bmatrix} f_{s,A'}(t)\\ g_{s,A'}(t)\\ 0\end{bmatrix};\quad t\in[0,2\pi],$$

Using again the properties of rotations, we claim that by rotating the point E', through angle ϕ around x–axis, we get point E on surface locus Δ. Analogous result can be stated if we rotate curve c around the y-axis.

3.1 Hyperboloid with two sheets

We apply Proposition 4 on finding the locus surface for a hyperboloid with two sheets. Let $A=(x_0,0,0)$, with $x_0<0$. For given a and b, we consider the surface generated by rotating the following hyperbola around the x–axis

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Following the calculations in [9], it can be shown that for fixed point $A'=(x_0,0)$ and point C' moving on the right branch of the hyperbola, that is,

$$C'=(a\cosh(t),b\sinh(t)),\quad t\in\mathbb{R},$$

the parametric equation of the locus curve, generated by point $E' = sC' + (1-s)D'$, is

$$d(t) = \begin{bmatrix} s\,a\cosh(t) + (s-1)\left(\frac{2a^2x_0\sinh(t)^2}{x_0^2-2a\cosh(t)x_0+a^2} + a\cosh(t)\right) \\ \\ s\,b\sinh(t) + (s-1)\frac{b\sinh(t)}{a\cosh(t)-x_0}\left(\frac{2a^2x_0\sinh(t)^2}{(x_0^2-2a\cosh(t)x_0+a^2)}a\cosh(t) + x_0\right) \end{bmatrix}, t \in \mathbb{R}.$$

It follows from Proposition 4 that the locus surface generated by point $E = sC + (1-s)D$ is

$$\Delta(u,v) = \begin{bmatrix} s\,a\cosh(u) + (s-1)\left(\frac{2a^2x_0\sinh(u)^2}{x_0^2-2a\cosh(u)x_0+a^2} + a\cosh(u)\right) \\ \\ \left(s\,b\sinh(u) + (s-1)\frac{b\sinh(u)}{a\cosh(u)-x_0}\left(\frac{2a^2x_0\sinh(t)^2}{(x_0^2-2a\cosh(t)x_0+a^2)}a\cosh(t) + x_0\right)\right)\cos(v) \\ \\ \left(s\,b\sinh(u) + (s-1)\frac{b\sinh(u)}{a\cosh(u)-x_0}\left(\frac{2a^2x_0\sinh(t)^2}{(x_0^2-2a\cosh(t)x_0+a^2)}a\cosh(t) + x_0\right)\right)\sin(v) \end{bmatrix},$$

where $u \in \mathbb{R},\ v \in [0,\pi]$.

Remark: We remark that the locus surface $\Delta(u,v)$ can be also obtained directly by generalizing the 2D techniques used in [9] or Section 2.1 to this 3D case (see Exploration [S4]). We demonstrate this effect by considering the following

Example 5 *Consider the surface generated by rotation around x–axis of the hyperbola*

$$\frac{x^2}{25} - \frac{4y^2}{81} = 1.$$

A locus trace and the locus surface generated by point $E = sC + (1-s)D$ when parameter $s = 0.1$, and the fixed point $A = (-1,0,0)$ are shown in Figures 5(a) and 5(b) respectively.

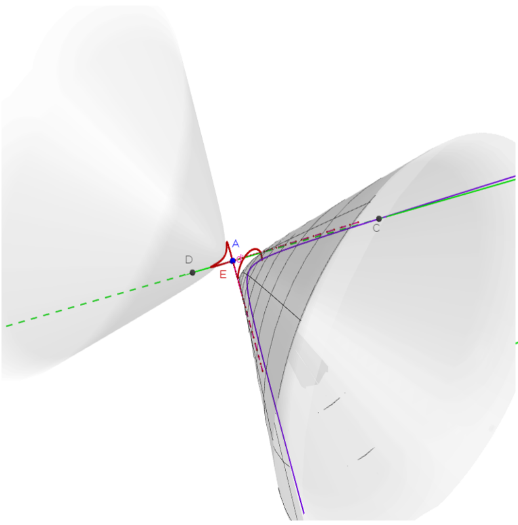

Figure 5(a). Some traces for the locus surface

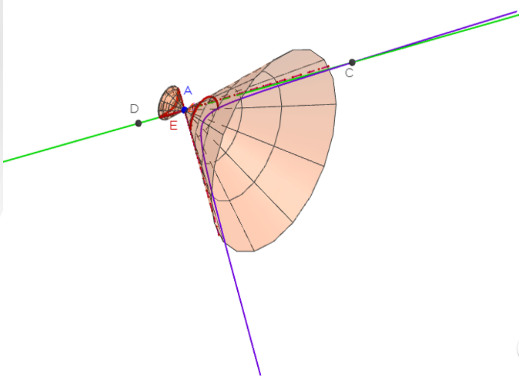

Figure 5(b). Locus surface

To explore the locus surface and reproduce an animation of the locus trace shown above see Exploration [S5].

3.2 Rotation surface generated by a cardioid

Let $A = (0, 0, 0)$ and $r(t) = a - a\cos t$. We consider the surface generated by rotating the following cardioid curve around the x-axis.

$$c(t) = (r(t)\cos(t), r(t)\sin(t)), \quad t \in [0, 2\pi]$$

We extend the derivation shown in [9] when $a = 1$. It can be shown that for the fixed point $A' = (0, 0)$, the parametric equation of the locus curve generated by point $E' = sC' + (1-s)D'$ is

$$d(t) = \big(a\cos(t)(\cos(t) - 2s + 1), a\sin(t)(s\cos(t) - 2\cos(t) + s)\big); \quad t \in [0, 2\pi].$$

If we agree that D must be different from A, it follows from Proposition 4 that the locus surface generated by point $E = sC + (1-s)D$ is

$$\Delta(u,v) = \begin{bmatrix} a\cos(u)\big(\cos(u) - 2s + 1\big) \\ a\sin(u)\big(s\cos(u) - 2\cos(u) + s\big)\cos(v) \\ a\sin(u)\big(s\cos(u) - 2\cos(u) + s\big)\sin(v) \end{bmatrix}; \quad u \in [0, 2\pi],\ v \in [0, \pi].$$

Example 6 *Consider the cardioidal surface generated by rotation around x–axis of the cardioid curve when $r(t) = 2 - 2\cos t$, and*

$$c(t) = (r(t)\cos(t), r(t)\sin(t)), \quad t \in [0, 2\pi].$$

Fixed point $A = (0, 0, 0)$ to be the origin; some traces, generated by the point, $E = sC+(1-s)D$, with parameter $s = 0.3$, together with the original cardioidal surface are shown in Figure 6(a). The corresponding locus surface is shown in Figure 6(b). To experiment the locus traces, we refer readers to see [S7].

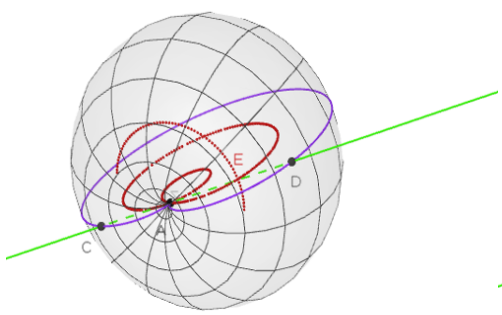

Figure 6(a). Some traces for the locus surface

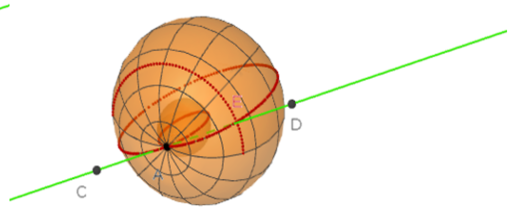

Figure 6(b). Locus surface

Explorations:

1. It is interesting to observe in this case that if $C' = (r(t)\cos(t), r(t)\sin(t))$ and the fixed point is at the origin $A' = (0,0)$, and take $D' = (r(t+\pi)\cos(t+\pi), r(t+\pi)\sin(t+\pi))$, then three points A', C' and D' are collinear, and the locus curve $E' = sC+(1-s)D$ is the same as the one obtained using the Vieta's formulas as shown in [9]. We may therefore apply the Proposition 4 to obtain the locus surface in 3D accordingly. For exploration on the locus E, please see [6]. We capture screen shot of the locus surface when $s = 0.7$ and the trace of $u = 0.88$ in yellow in Figure 6(c).

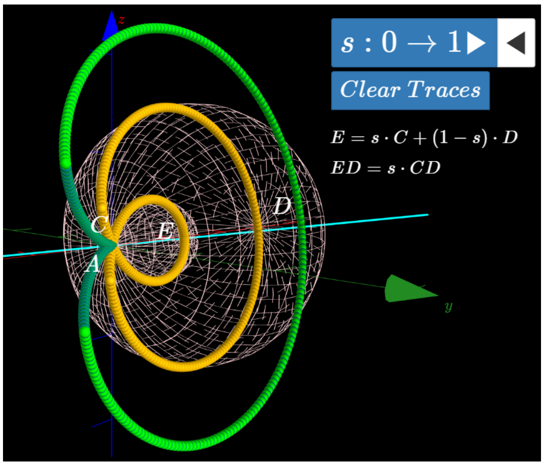

Figure 6(c) Locus surface when
$s = 0.7$, trace of $v = 4.88$ (yellow).

(13)

2. We shall discuss in details how we apply the Vieta's formulas directly, without using the rotation technique, when finding the locus surface for the 3D cardioidal surface, in the next subsection.

3. We remark that if $C' = (r(t)\cos(t), r(t)\sin(t))$ and the fixed point is at the origin $A' = (0,0)$, then the antipodal point with respective to A turns out to be $D' = (r(t+\pi)\cos(t+\pi), r(t+\pi)\sin(t+\pi))$. We then apply the Proposition 4 to obtain the corresponding locus surface in 3D accordingly.

3.3 Locus of 2D or 3D with a higher degree polynomial

We shall investigate in some special cases where higher degree polynomial equations can be reduced to quadratic equations when finding roots. Thus the Vieta's formulas can be applied. We recall that when the line l, connecting the fixed point A and a given point C on a surface, it is possible that l becomes a vertical at a specified t. In such a case, we call t to be a singular point. We use the preceding example to demonstrate how we can ignore the singular point so the Vieta's formulas can be applied. Consider $c(u) = (r(u)\cos u, r(u)\sin u)$, where $r(u) = 2 - 2\cos u$, and $u \in [0, 2\pi]$. We discuss the following two scenarios:

Case 1. When $A = (0, 0, 0)$. We want to find the locus surface $E = sC + (1 - s)D$. We outline the procedure as follows:

1. Since implicit equation for $c(u)$ is
$$x^4 + 2y^2x^2 + y^4 + 4x^3 + 4xy^2 - 4y^2 = 0,$$
the 3D implicit equation for the cardioidal surface Σ by rotating $c(t)$ around the x-axis is
$$x^4 + 2(y^2 + z^2)x^2 + (y^2 + z^2)^2 + 4x^3 + 4x(y^2 + z^2) - 4(y^2 + z^2) = 0.$$

2. The parametric equation for the cardioidal surface Σ is
$$\begin{bmatrix} (2 - 2\cos u)\cos u \\ (2 - 2\cos u)\sin u \cos v \\ (2 - 2\cos u)\sin u \sin v \end{bmatrix}.$$
We adopt the techniques from Eqs 4 – 6, to get the x–coordinate of D. The polynomial equation $p(x) = 0$ turns out to be a degree 4 with double roots at $x = 0$. Thus, $\frac{p(x)}{x^2}$ is quadratic as shown below:
$$\frac{p(x)}{x^2} = \frac{(x^2 + (4x - 4)(\cos u)^2 + 4\cos^4 u)}{(\cos u)^4}.$$

3. We ignore the singular point when $u = \frac{\pi}{2}$ and apply the Vieta's formulas, so we obtain the coordinates of point D and the locus surface $\Delta(u, v)$ satisfying $E = sC + (1 - s)D$ accordingly. We depict the locus surface when $s = 0.3$ using Maple [3] in Figure 7.

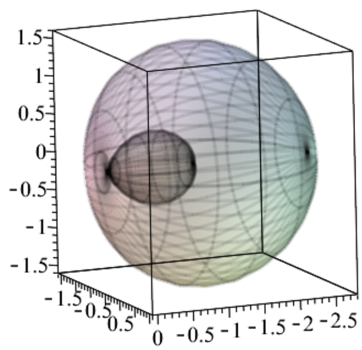

Figure 7. Locus surface when $s = 0.3$

4. For exploration with Maple [3], see [S6].

Case 2. When $A = (-4, 0, 0)$. Find the locus surface $E = sC + (1 - s)D$ for the surface Σ. With a DGS in hand (such as [2]), it is easy to see the following observations for the 2D case of $c(u) = (r(u)\cos u, r(u)\sin u)$.

1. The point D' is fixed at $A' = (-4, 0)$ and when $C' \in c(u)$. We see the line $A'C'$ becomes a vertical line when $t = \pi$, where the singular point is located.
2. We follow the Vieta's techniques used in [9] or follow the ideas described in Section 2.1, we find an equation of degree four polynomial p in x with the coefficient of $(r(u)\cos u + 4)^n$ in the denominator, where $n = 1, 2, 3, 4$.
3. We avoid the singular point when $r(u)\cos u + 4 = 0$ or $u = \pi$, by multiplying $(r(u)\cos u + 4)^4$ by p, to obtain another degree 4 polynomial in x, say q.
4. We solve $q(x) = 0$ and yield four solutions, which are two real roots and two complex roots. We ignore those two complex solutions, and the real solutions as expected to be -4 and $r(u)\cos u$ respectively.
5. Therefore, as expected $D = A' = (-4, 0)$, and we find the locus $E = sC + (1 - s)D$ to be

$$\begin{bmatrix} x_e(t) \\ y_e(t) \end{bmatrix} = \begin{bmatrix} s\,(r\cos t) - 4(1 - s) \\ s\,(r\sin t) \end{bmatrix}.$$

Accordingly, we can find the respective locus surface for the cardioidal surface Σ by rotating $\begin{bmatrix} x_e(t) \\ y_e(t) \end{bmatrix}$ around the $x-$axis. The next exercise involves an equation of degree 6 polynomial and yet the Vieta's formulae are still applicable because the equation of degree 6 polynomial has quadruple repeated roots at the origin.

Exercise 7 *We consider* $r(t) = -2\cos t + 2\cos^2 t$ *and* $c(t) = [r(t)\cos t, r(t)\sin t]$, *where* $t \in [0, 2\pi]$, *see Figure 8(a). Let the fixed point A be at the origin. Find the locus surface:*

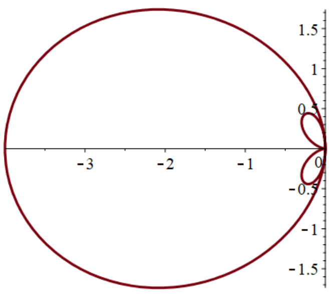

Figure 8(a).
$r(t) = -2\cos t + 2\cos^2 t$

Method 1. Thanks to [1], we obtain the implicit equation for $c(t)$ to be

$$x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 4x^5 + 8x^3y^2 + 4xy^4 + 4x^2y^2 = 0.$$

We invite readers to investigate that the polynomial $p(x)$ while finding the point D (when setting $p(x) = 0$) turns out to be a degree 6 with four multiple roots at the origin $(0, 0, 0)$. Thus the Vieta's formula can be applied again. [See Maple worksheet S8].

Method 2. We invite readers to apply $C = (r(t)\cos t, r(t)\sin t)$ and its antipodal point, $D' = (r(t+\pi)\cos(t+\pi), r(t+\pi)\sin(t+\pi))$ when the fixed point A is at the origin. We then apply the Proposition 4 to obtain the corresponding locus surface in 3D accordingly.

Explorations: Here is another Exercise that the Vieta's formulae can be applied directly to find its 2D locus curve.

1. Let $r(t) = -2\cos t + 2\cos^2 t$ and $c(t) = [r(t)\cos t, r(t)\sin t]$, where $t \in [0, 2\pi]$, see Figure 8(b).

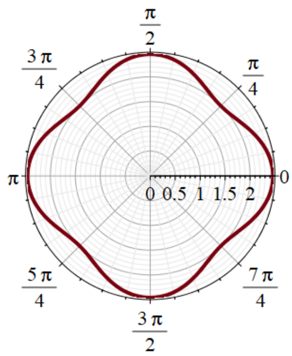

Figure 8(b). Graph of $r(t) = -2\cos t + 2\cos^2 t$

(14)

We obtain the implicit equation for $c(t)$ to be $x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 4x^5 + 8x^3y^2 + 4xy^4 + 4x^2y^2 = 0$ (thanks to [1]). If the fixed point is at $A = (0, 0, 0)$, then the locus $E' = sC' + (1-s)D'$ can be found by using Vieta's formulae. Consequently, if we rotate $c(t)$ around the $x\!-\!axis$ to obtain a 3D surface, Vieta's formulae can be applied directly on the implicit equation, $x^6 + 3x^4(y^2+z^2) + 3x^2(y^2+z^2)^2 + (y^2+z^2)^3 + 4x^5 + 8x^3(y^2+z^2) + 4x(y^2+z^2)^2 + 4x^2(y^2+z^2) = 0$, to find the locus surface $E = sC + (1-s)D$.

2. Alternatively, one can apply the Method 2 in the preceding Exercise to obtain the 2D locus curve and 3D locus surface accordingly.

3. As explained, suppose the fixed point is at the origin $A' = (0, 0)$, $C' = (r(t)\cos(t), r(t)\sin(t))$, then we may use $D' = (r(t+\pi)\cos(t+\pi), r(t+\pi)\sin(t+\pi))$ to find its locus curve. (We call such method as antipodal method). It is trivial that if we let C_A to be the class of curves where we can apply the antipodal method to find the carbon-dating locus curves, and we let C_V to be the class of curves where we can apply the Vieta's formulae to find the corresponding locus curves, then we see $C_V \subsetneq C_A$. However, the Vieta's formulae can be tried when the fixed point is not at the origin.

4 Locus Surfaces For Central Symmetric Surfaces

The key of finding the locus curve or surfaces is to find the roots of the implicit equation that results when we consider the intersection between the line, AC, and the given surface. We have discussed how Vieta's formulas can be used in calculating roots for quadratic or special higher degree polynomial equations. The existences of such polynomials require us to know the corresponding implicit equations for the given curves or surfaces in advance. In this section, we consider special surfaces, other than spheres, that are central symmetric (symmetric to the origin). As a result, we see $D = -C$ for central symmetric surfaces, and the corresponding locus surfaces can be found easily and we don't need its implicit polynomial equation. The Roman surface we shall explore in this section is characterized by a stronger property, namely $A = C = D$, and the locus surface will be therefore a fixed surface. First, we consider the following

Proposition 8 *If Σ is symmetric with respect to the origin $(0,0,0)$, then the locus surface Δ is obtained by contracting (dilating) surface Σ by $|2s-1|$.*

Proof

Let E be a point on surface Δ. By construction, there are points C and D on the surface Σ such that $E = sC + (1-s)D$. Since Σ is symmetric with respect fixed point A, point D must be "antipode" of point C, so $D = -C$ and therefore

$$E = sC + (1-s)D = (2s-1)C$$

that is, $\Delta \subset (2s-1)\Sigma$. The reciprocal inclusion is analogous.

4.1 Cyclotomic surface

Let $A = (0,0,0)$. A cyclotomic surface can be defined as

$$\Sigma = \left\{(x,y,z) \in \mathbb{R}^3 : \left(x^2+y^2+z^2\right)\left(x^2+y^2\right) - a^2x^2 = 0\right\}$$

It is clear that Σ is symmetric with respect to A by its algebraic implicit equation defined above. Using parametrization for Σ, we can also represent a generic point C on Σ as

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} r(u)\cos(u)\sin(v) \\ r(u)\sin(u)\sin(v) \\ r(u)\cos(v) \end{bmatrix},$$

where $r(u) = a\cos u$, $u \in [0, 2\pi]$ is some kind of "variable longitude", and $v \in [-\pi, \pi]$ is some kind of "extended colatitude". Along this observation, we can view cyclotomic surface as an equation in the spherical coordinate (ρ, u, v) as follows:

$$\rho = r(u),$$

with $u \in [0, 2\pi]$ and $v \in [-\pi, \pi]$. We see that the spherical surface is clearly central symmetric. As a result, the intersection point between the line AC and the surface Σ has to be at $D = -C$

without knowing its implicit equation. Hence, it follows from Proposition 8 that the locus surface generated by point $E = sC + (1-s)D$ is

$$\Delta(u,v) = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = |2s-1| \begin{bmatrix} r(u)\cos(u)\sin(v) \\ r(u)\sin(u)\sin(v) \\ r(u)\cos(v) \end{bmatrix}; \; u \in [0,2\pi] \text{ and } v \in [-\pi,\pi].$$

We present the two following examples:

Example 9 *Consider the cyclotomic surface*

$$\left(x^2+y^2+z^2\right)\left(x^2+y^2\right) - 4x^2 = 0,$$

or the spherical equation of $\rho = 2\cos u$, with $u \in [0,2\pi]$ and $v \in [-\pi,\pi]$, see Figure 9.

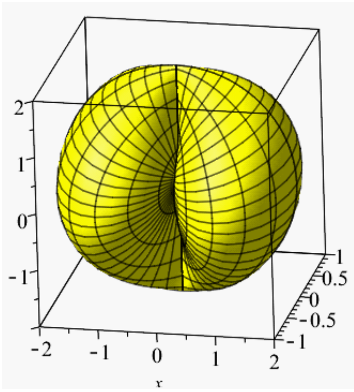

Figure 9. Spherical surface of $\rho = 2\cos u$

A locus trace with the original surface and the locus surface,when parameter $s = 0.25$ and fixed point $A = (0,0,0)$, are shown respectively in Figures 10(a) and 10(b) respectively..

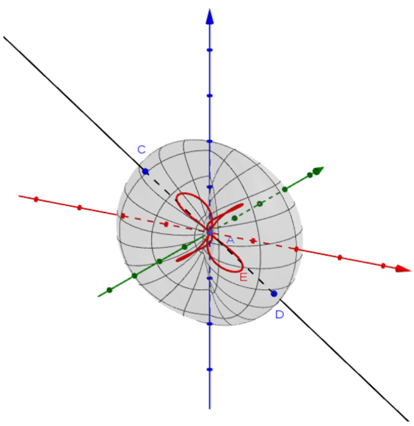

Figure 10(a) Trace and the surface

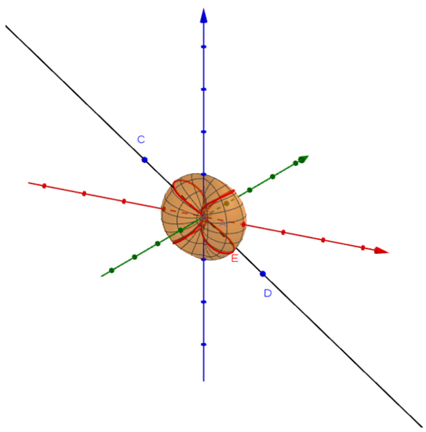

Figure 10(b). Locus surface with a trace

To reproduce an animation of the locus trace shown above, see Exploration [S9]. We next extend the cyclotomic surface by varying its spherical equation and note that the locus surface is obtainable without knowing its implicit equation.

Example 10 *We consider the surface generated by the spherical equation of* $\rho = r(u)\cos u$, *with* $r(u) = 2 - 2\cos u, u \in [0, 2\pi]$ *and* $v \in [-\pi, \pi]$ *. we see the surface to be central symmetric, and thus the behavior of the corresponding locus surface follows the Proposition 8. We observe the following key points:*

1. As we have observed that a surface represented by $\rho = f(u)$ will be central symmetric. We note the polar plot for $r(u) = 2 - 2\cos u, u \in [0, 2\pi]$ is shown in Figure 11(a).

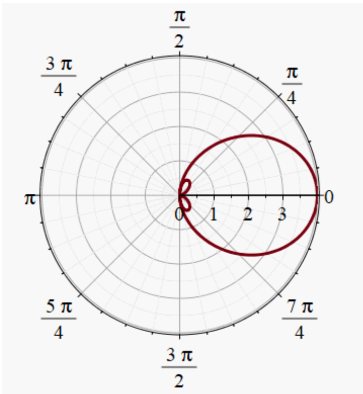

Figure 11(a). $r(u) = 2 - 2\cos u, u \in [0, 2\pi]$

2. The spherical plot for $\rho = r(u)\cos u$, together with two respective slices when $v = \frac{\pi}{2}$ and $v = -\frac{\pi}{2}$ when $u \in [0, 2\pi]$, is shown below in Figure 11(b).

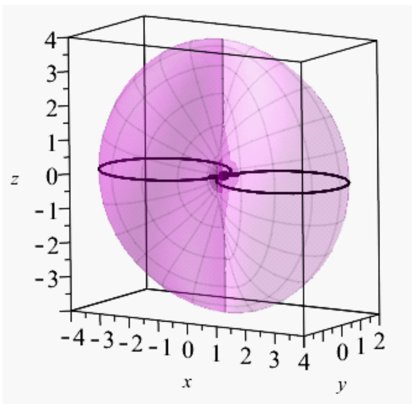

Figure 11(b). Traces for $\rho = r(u)\cos u, v = \frac{\pi}{2}$ and $v = -\frac{\pi}{2}$

3. The spherical plot for $\rho = r(u)\cos u$, together with two respective slices when $v = \frac{\pi}{4}$ and $v = -\frac{3\pi}{4}$ when $u \in [0, 2\pi]$, is shown below in Figure 11(c).

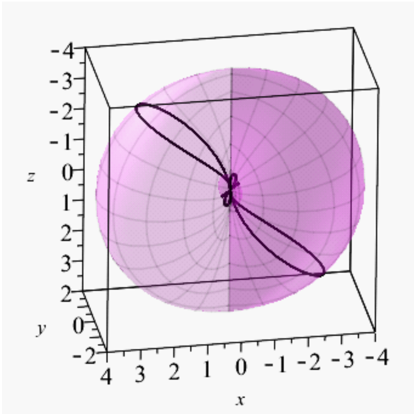

Figure 12(c). Traces for $\rho = r(u)\cos u, v = \frac{\pi}{4}$ and $v = -\frac{3\pi}{4}$

4. We can find the locus surface for a central symmetric surface without knowing its implicit equation.

5 The locus surface is fixed

We start with simple 2D cases where the locus curves stay as fixed curves. In particular, we discuss a 2D scenario where the antipodal point D happens to be the same as the moving point C when the fixed point is at the origin.

Example 11 *We consider* $r(t) = \sin t \cos 2t, t \in [0, 2\pi]$, *see Figure 13. The implicit equation for* $r(t)$ *involves a polynomial of degree 4. It is easy to verify that the degree 4 implicit equation,* $x^4 + 2x^2y^2 + y^4 - x^2y + y^3 = 0$, *produces triple roots at the origin and consequently, the required point* D *is the same as the moving point* C*. Thus, the locus for* $E = sC + (1-s)D$ *becomes* *the point D or C. We leave this to readers to explore.*

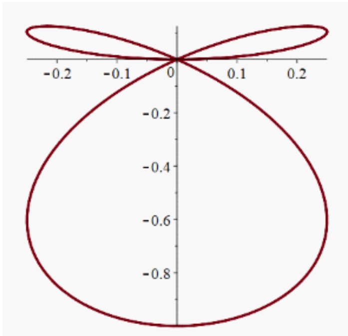

Figure 13. $r(t) = \sin t \cos 2t, t \in [0, 2\pi]$

Remarks:

- Indeed there are cases when the locus is the same as its original curve when the fixed point is at the origin. Readers can also try polar curve of $r = (\cos t)^3, t \in [0, 2\pi]$, where its implicit equation is $-x^3 + x^4 + 2x^2y^2 + y^4 = 0$.
- It is trivial that if the polynomial equation $p(x) = 0$ (degree of $p(x)$ is 4), while finding the point D produces triple roots at the origin, then the antipodal points $D = C$ with respective to the origin.

Now we explore the Roman or Steiner surface (see [8]) in 3D, where we shall see that the locus surface stays fixed too. We extract the following observations which are related to our paper as follows:

1. We can view the Roman surface as the image of a sphere of radius r, centered at the origin under the map of $f(x, y, z) = (yz, xz, xy)$, which gives a degree four implicit formula:

$$x^2y^2 + y^2z^2 + z^2x^2 - r^2xyz = 0.$$

The corresponding parametric equation for the Roman surface can be written as follows:

$$\begin{aligned} x &= r^2 \cos u \cos v \sin v \\ y &= r^2 \sin u \cos v \sin v \\ z &= r^2 \cos u \sin u \cos^2 v, \end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [0, \pi]$ .The Roman surface when $r = 1$ is shown in Figure 14.

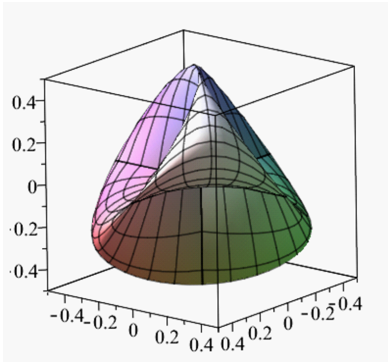

Figure 14. Roman surface
$x^2y^2 + y^2z^2 + z^2x^2 - xyz = 0$

2. We see if the fixed point $A = (0, 0, 0)$, and $C = (x, y, z)$ is on the Roman surface, then its antipodal point $D = (-x, -y, -z)$ is the same as $C = (x, y, z)$. This leads to the locus surface E based on $E = sC + (1 - s)D$ to be a fixed surface, unchanged from the original Roman surface.

3. We can also follow the derivation of obtaining the locus in Section 2 that the locus surface for the Roman surface is indeed fixed, with the help of its implicit equation. The key step is that it involves an equation of a degree four quartic polynomial in x, which has three triple zeros at $x = 0$. Consequently, it leads to $D = C$. We leave this simple exercise to the readers to verify.

By looking at problems discussed in Section 3.3 and these two special surfaces discussed above, it will be interesting area in algebraic geometry to see if we categorize those surfaces of higher orders, that has the property of any one of the following properties: (a) the fixed point $A = D$, (b) $C = -D$, or (c) $C = D$.

6 Conclusion

In this paper, we explored 3D generalization of the locus problems discussed in ([9]) and beyond. We have seen situations where Vieta's formulas can be adopted directly for quadric surfaces. We have also seen the Vieta's formulas can be applied on some special higher degree polynomial equations. In view of Example 6 of $r = a - a\cos t$, we have seen that it is possible to find the locus curve when the fixed point A is at the origin, $(0, 0)$ or at $(-2a, 0)$. However, when the fixed point $A = (x, 0)$ is an interior point, where $x \in (-2a, 0)$, then we will be led into finding solutions for polynomial equations of degree higher than two, which involve real and complex roots Therefore, we will investigate the followings further in our future work:

1. We will explore problems that involve in extracting proper real roots to assist us finding our locus accordingly.

2. We will investigate cases when the point D does not posses a closed form but only in numerical form. We shall see how the numerical approximation value D can assist us finding the corresponding locus E.

Moreover, if C and D are both on the given surface Σ, and if we vary $s \in [0,1]$, then E, described in our Main problem, defines a transformation between C and D (when $s = 0$, $E = D$ and if $s = 1$, $E = C$). On the other hand, if we are given two surfaces, Σ and Σ', with $C \in \Sigma$ and $D \in \Sigma'$, where C and D are still connected by a line l through a fixed point A. Then the locus E, in this case, can be viewed as a transformation as a function of s, between two topological surfaces Σ and Σ'. Another area we will explore in our future work is the following scenario: For a given by Σ and a fixed point at A, by selecting three vectors, AB, AC and AD, where B, C and D are on Σ respectively. We explore the new generated affine surface of

$$\Sigma' = r \cdot AB + s \cdot AC + t \cdot AD,$$

where $r, s, t \in \mathbb{R}$ with $r+s+t = 1$. Therefore, we may extend the areas we discuss in this paper to problems and applications in topology, algebraic geometry, computer graphics, projective geometry and etc. with the help of technological tools.

7 Acknowledgements

We would like to thank Dr. Harald Pleym for advising on making some Maple animations more robust and interesting. We also thank anonymous reviewers for making constructive comments to improve the paper.

8 Supplementary Electronic Materials

[S1] `s1-Section2.1.wxm` (wxMaxima worksheet for Section 2.1).

[S2] `s2-Section2.2.wxm` (wxMaxima worksheet for Section 2.2).

[S3] `s3-Section2.ggb` (GeoGebra worksheet for Section 2).

[S4] `s4-Section3.1.mw` (Maple worksheet for Section 3.1).

[S5] `s5-Section3.1.ggb` GeoGebra worksheet for Section 3.1)

[S6] `s6-Section3.3.mw` (Maple worksheet for Section 3.2).

[S7] `s7-Section3.2.ggb` (GeoGebra worksheet for Section 3.2).

[S8] `s8-Exercise8.mw` (Maple worksheet for Section 3.3).

[S9] `s9-Section4.1.ggb` (GeoGebra worksheet for Section 4.1)

References

[1] Geometry Expressions, see http://www.geometryexpressions.com/.

[2] GeoGebra (release 6.0.562 / October 2019), see https://www.geogebra.org/.

[3] Maple, A product of Maplesoft, see http://Maplesoft.com/.

[4] maxima (release 5.43.0 / May 2019), see http://maxima.sourceforge.net/.

[5] Netpad: A web-based interactive geometry software from China: https://www.netpad.net.cn/resource_web/course/#/202364.

[6] Netpad: https://www.netpad.net.cn/resource_web/course/#/222555.

[7] Shi Gi Gin Bang "Strategies for High School Mathematics Complete Review" by Cuiun Guan Zhiming. Century Gold. Yanbian University Press, 2015.

[8] The Roman Surface, Wikipedia, https://en.wikipedia.org/wiki/Roman_surface.

[9] Yang, W.-C., Locus Resulted From Lines Passing Through A Fixed Point And A Closed Curve, *The Electronic Journal of Mathematics and Technology*, Volume 14, Number 1, pp. 1-17, 2020.